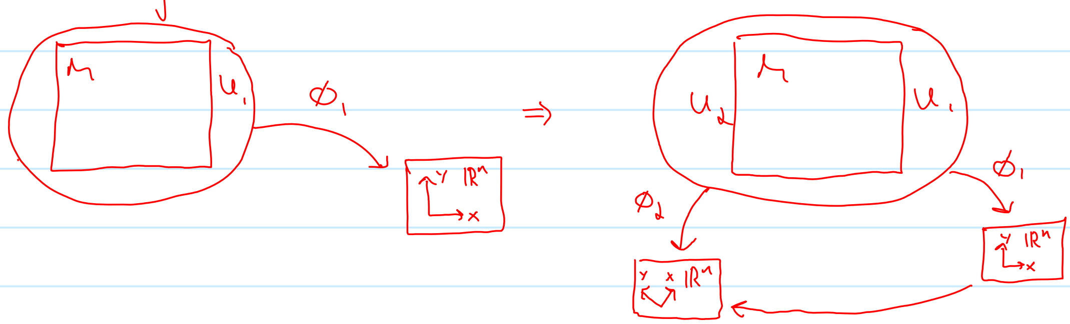
So far in our discussion of manifolds we know that we can construct an atlas of charts $\{(U_\alpha, \phi_\alpha)\}$ and use the maps $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ to coordinatize M (on each chart and then sew the patches together w/ transition functions, i.e. coord. changes).

One important thing to remember is that we cannot always cover M w/ one chart, hence we cannot have one "global" coordinate system on M. If we can cover M w/ one chart then we <u>can</u> (if we want) set up a global coordinate system.

Okay, so we use coordinate transformations to piece together patches, but what about a coordinate transformation for a global set of coordinates?

Example: $M = \mathbb{R}^n$



$\phi_2 \circ \phi_1^{-1} \equiv \phi_{21}$

That is, we just use two charts covering M but equipped w/ different maps!

Okay, coordinate changes... nuff said!

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If we know how coordinate changes happen, can we say anything about derivatives w.r.t. coordinates?

there are 4 expressions here in 4D

A coordinate change gives: $\underset{\text{new}}{x^{\mu'}}(\underset{\text{as function of old}}{x^{\mu}})$

Consider the chain rule: $\underbrace{\frac{d}{dx}\phi(g(x))}_{\text{secretly } \frac{d}{dx}(\phi\circ g)} = \frac{d\phi}{dg}\frac{dg}{dx}$

Now bump it up: $g_1(x,y)$ $g_2(x,y)$ $\phi(g_1, g_2)$

$$\Rightarrow \begin{aligned} \frac{\partial\phi}{\partial x} &= \frac{\partial\phi}{\partial g_1}\frac{\partial g_1}{\partial x} + \frac{\partial\phi}{\partial g_2}\frac{\partial g_2}{\partial x} \\ \frac{\partial\phi}{\partial y} &= \frac{\partial\phi}{\partial g_1}\frac{\partial g_1}{\partial y} + \frac{\partial\phi}{\partial g_2}\frac{\partial g_2}{\partial y} \end{aligned}$$

$$\Downarrow$$

$$\frac{\partial}{\partial x^{\mu}}\phi(\underbrace{g^{\lambda}}_{g^{\lambda}(x^{\mu})}) = \frac{\partial\phi}{\partial g^{\lambda}}\frac{\partial g^{\lambda}}{\partial x^{\mu}}$$

Now think of g as a coordinate change: $g^{\mu'}(x^{\mu})$ or $x^{\mu'}(x^{\mu})$

Then: $\frac{\partial}{\partial x^{\mu}}\phi(x^{\mu'}) = \frac{\partial\phi}{\partial x^{\mu'}}\frac{\partial x^{\mu'}}{\partial x^{\mu}} \Rightarrow \frac{\partial}{\partial x^{\mu}} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial}{\partial x^{\mu'}}$

or $\frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial}{\partial x^{\mu}}$ Transformation Law for partial derivatives

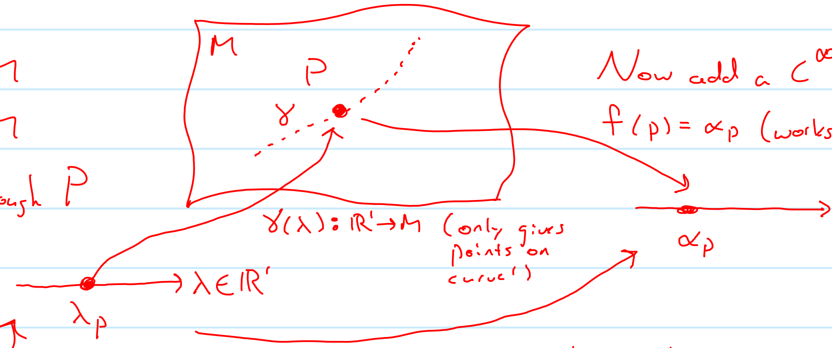
What about more complicated objects like vectors, etc.?

We know that these objects live in tangent & cotangent spaces, so let's get a formal definition of these. First off, they should be well defined even w/out introducing coordinates (that is an option).

Begin w/ Manifold M

Point $P \in M$

Curve γ through P



Now add a C^∞ map $f: M \to \mathbb{R}^1$

$f(p) = \alpha_p$ (works for any $P \in M$!)

This is not a chart!

$f \circ \gamma: \mathbb{R}^1 \to \mathbb{R}^1$

that is $f \circ \gamma(\lambda_p) = \alpha_p$

In words: Changing λ changes p which changes α. So we can consider $\underbrace{\frac{df}{d\lambda} \equiv \frac{d}{d\lambda}(f \circ \gamma)}$

The directional derivative of f along λ (which is tangent to the curve γ).

If we now consider all curves through P  parameterized by $(\lambda, \alpha, \beta, \text{etc.})$

then the set of directional derivatives for these curves forms a vector space $\left\{\frac{d}{d\lambda}, \frac{d}{d\alpha}, \frac{d}{d\beta}, \text{etc.}\right\}$

Recall that a vector space satisfies:

a) $V + W = Y$ closure under $+$

b) $aV = Y$ closure under scalar $\times$

c) associativity of $+$

d) commutativity of $+$

e) identity in $+$

f) inverse in $+$

g, h, i, ...) rules for scalar $\times$

So we have established a vector space of tangents to M at P, i.e. the tangent space at P!

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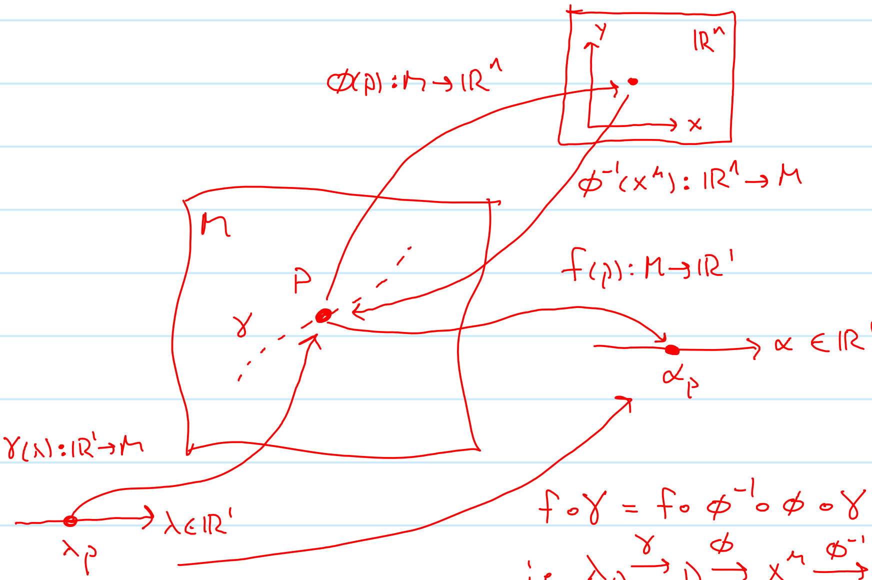
Now that we have a tangent space, let's see if we can make use of the charts (and coordinates) to setup a basis and thus coordinate representations of vectors. Then we can ask how components transform under coordinate transformations.



$f \circ \gamma = f \circ \phi^{-1} \circ \phi \circ \gamma : \lambda_p \to \alpha_p$

i.e. $\lambda_p \xrightarrow{\gamma} p \xrightarrow{\phi} x^\mu \xrightarrow{\phi^{-1}} p \xrightarrow{f} \alpha_p$

Here comes the magic!

$$\frac{df}{d\lambda} = \frac{d}{d\lambda}\left(f \circ \phi^{-1} \circ \phi \circ \gamma\right)$$

$\phi \circ \gamma : \mathbb{R}^1 \to \mathbb{R}^n$ so we can label it $(\phi \circ \gamma)^\mu$

$$= \frac{\partial (f \circ \phi^{-1})}{\partial (\phi \circ \gamma)^\mu} \frac{d(\phi \circ \gamma)^\mu}{d\lambda}$$ using the chain rule on $f \circ \phi^{-1}(\phi \circ \gamma)$

Since: $(\phi \circ \gamma)^\mu : \mathbb{R} \to \mathbb{R}^n$ we can call it $x^\mu(\lambda)$

Since: $(f \circ \phi^{-1}) : \mathbb{R}^n \to \mathbb{R}$ we can call it $f(x^\mu)$

Then: $\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu} \frac{dx^\mu}{d\lambda}$ components

But f is arbitrary so: $\underbrace{\frac{d}{d\lambda}}_{\text{any directional derivative, i.e. tangent vector}} = \frac{dx^\mu}{d\lambda} \partial_\mu$ basis vectors in <u>tangent space</u>

We have established what is called a coordinate-adapted basis in the tangent space.

But wait... any vector V must live in the tangent space, and so must be expressible as $V = V^{\mu}\partial_{\mu}$.

It gets better because we know that V is invariant under coordinate changes and $\partial_{\mu} \to \partial_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\partial_{\mu}$

So we can infer that: $V^{\mu}\partial_{\mu} = V^{\mu'}\partial_{\mu'}$

$$= V^{\mu'}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\partial_{\mu}$$

$$\Downarrow$$

$$= V^{\nu}\underbrace{\frac{\partial x^{\mu'}}{\partial x^{\nu}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}}_{\mathbb{I}}\partial_{\mu}$$

Thus: $V^{\mu} \to V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}V^{\mu}$ Transformation law for vector components.

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Dual Vectors

We can now proceed much like what we did in SR.

$W = \omega_\mu \hat{\theta}^{(\mu)}$ where $\hat{\theta}^{(\mu)} \underbrace{\hat{e}_{(\nu)}}_{\text{We know these are } \partial_\nu} = \delta^\mu{}_\nu$

We will call these dx^μ

Then: $dx^\mu \partial_\nu = dx^{\mu'} \partial_{\nu'} \Rightarrow dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$ Transformation law for dual basis vectors

And: $\omega_\mu \rightarrow \omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu$ Transformation law for dual vector components.

Tensors

No suprise here: $\underbrace{T^{\mu\nu}{}_{\alpha\beta} \rightarrow T^{\mu'\nu'}{}_{\alpha'\beta'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^\beta}{\partial x^{\beta'}} T^{\mu\nu}{}_{\alpha\beta}}_{\text{Transformation law for tensor components}}$